The music industry is often seen as the primary victim of piracy. It's long term reduction in profit has been attributed to the large number of pirates whose actions harm both artists and their representatives\citep{B03}. More recent analysis estimates that as many copies of popular music is being pirated as is being accessed through legal means\citep{O15}.

Still, the damage caused by this sort of activity is often ambiguous and hard to compute due to the fact that some artists choose to give their work for free. Today, music is often uploaded for free on public platforms such as youtube, in the music industry this may be because the revenue potential of concerts would rise, in our model we model this sort of indirect revenue as a complementary good. Nevertheless, there also exist strong movements against piracy, even though though it is known that the cost of policing such individual behavior is often so high, both for individual firms, and for the state.

Other goods may follow similar rules of rewards which don’t come directly with one instance of interaction with the consumer. Televised series for instance are all about reputation, the owner of the series will gain a boost in reputation and be able to charge higher prices for its advertising slots. The value of a televised series to a consumer is not only the direct experience of watching it but also the socialization that follows it afterward. This is also true of things such as spectator sports, where a more popular following will imply that the associated organizations will be able to charge higher prices for events or membership. It is hard to imagine that the world cup attracts the number of viewers it does because a large portion of the population is interested in the intricacies of a good match, instead this is likely a communal event. The commonly employed example of the QWERTY keyboard is also relevant here but this type of analysis extends to other things such as the consumption of current events.

Another example that may shed light is software. Statistical packages in general are software products which have the value which is directly provided by the firm and the value which leaks from other consumers and is generally dependent on the number of users, this is because the number of packages and versatility of the platforms depends on their respective user base. This partly explains the rise of open source software, with Python, R, their active user base produces packages which increases the value of these platforms. However, this is not to say that proprietary software does not gain value from a richer user base, for instance, STATA hosts events for licensees.

On a more philosophical note, all value derived from goods can be separated into intrinsic and extrinsic value. The intrinsic value is the part of the value which is about the good in itself, independent of other people’s consumption and valuation. The extrinsic portion of the value is that which is dependent on other people’s choices and uses.

In general, digital goods, once they are created, have trivial marginal cost to distribute and can essentially be distributed to everyone with a computer and access to the Internet. A firm trying to exploit this now existing good would do so by setting the price which will maximize profits, which would depend on how it perceives the distribution of the willingness to pay of consumer to be. In a world where the firm can perfectly price discriminate, it would charge each consumer exactly what their valuation would be and in a world where it would not be able to distinguish at all, it would use the distribution of the willingness to pay to separate the world into buyers and non-buyers. Piracy on the other hand, is somewhere between these two worlds.

Piracy is often dismissed as merely a clear loss for the firm, under this threat, they generally have two main options. Fundamentally, they can either attempt to increase the value of their bought product, or decrease piracy by increasing the potential cost of pirating. The ability to differentiate the bought product from the pirated product is a challenge which requires balancing costs. An example of this strategy in the entertainment industry is already seen through limited edition sets that include various extra content such as conceptual art or more information on the development process. In the case of sports it usually means higher pixelation or additional functionality such as the ability to pause and rewind games. Often companies structure themselves in a way as to offer a free good of base value and giving an improved product to those who pay. This added content is most often coveted by the consumers who have a higher willingness to pay.

The present situation can be framed as a choice between the relative level at which a firm will rely on the stick versus the carrot. Firms with copyright claims on their products have in their arsenal both a carrot and a stick. The carrot, in this case, is the ability to attract consumers by offering a high value product. In contrast, the stick is the ability to increase the cost of piracy; this would incite pirates to switch to buying.

Conceptually, the carrot is the improvement of the product but also a decreased price and, while improving the product is often costly, changing the price is not. This means that there is automatically a built-in mechanism to incite firms to rely on the stick. Nevertheless, with regards to copyright, firms may choose to incur some cost of chasing after individual consumers or distributor firms.

A company that decreases the cost of pirating can expect two types of effects. The first is that the consumers who would have bought the product will instead pirate it and similarly the consumers who would have neither pirated nor bought it may decide to obtain it through piracy. The relative importance of these two effects would likely depend on the level of differentiation between the pirated product and bought product.

The bought and pirated product, might differ in value naturally without extraordinary effort from the enterprise or government. For instance acquiring a product from a non-official digital source may entail some risk of downloading a virus or being hacked, this would be a natural level of a priori product degradation. Specific effects differentiating the socialization values between the two product can also be imagined. For instance a social stigma may cause the pirates to derive a lower proportion of value from socializing. Consumers may also derive additional socialization value from the bought product because it may be used as a signaling mechanism.

Data indicates that the consumer who will pirate will most likely be a consumer with a lower willingness to pay. The business software alliance estimates that the developing world has a much higher rate of piracy than the developed world, with countries like the United States, Japan, Luxembourg, New Zealand, Belgium, and Denmark having a piracy rate below $25$ per cent whilst countries like Bangladesh, Georgia, Armenia, Zimbabwe, Sri Lanka, Azerbaijan have rates exceeding $90$ per cent. \citep{BSA09}

In some domains these effects may be separate, that is, if there is an increase in the number of pirates, this has no effect on the number of people who will buy and vice versa. Under such a presumption, a firm would only be concerned with maximizing the number of buyers, meaning that it will set the cost of piracy as high as possible. However, we can imagine that there are some domains where an increase in the number of pirates has an indirect effect which increases the number of buyers.

To consider why it may be that an increase in the number of pirates may lead to an increase in the number of buyers we need only consider that the decision to purchase a good depends not only on the intrinsic value of the good but also on the number of consumers who are consuming it. This means that there is a secondary source of utility stemming from the communal aspect of a good. This can be interpreted as a sort of socialization utility.

If we gradually relax the assumption of utilities being independent we have what is called in the literature a 'network good'. It is possible to envision most goods as having an intrinsic value and an extrinsic value. This relative ratio of intrinsic to extrinsic value will likely vary substantially between cultures, space and time. The domain in which this is true is likely to extend much farther than what common intuition would entail. For instance, even perishable goods, demand for which is commonly thought of as inelastic, such as food, are not necessarily exempt from this feedback process. Accordingly, much of what can be deemed "group identity" can be represented within a network good framework. Within the framework of network goods, stealing and not consuming are not worth the same to the company.

This paper focuses on cases where firms effectively have a monopoly on the network good in question. This may be for various reasons, ranging from intellectual property rights to geographical advantages and leaves the model open for an interpretation where the profit is derived from a complementary good. The main contribution of this paper is a closed form solutions for the optimal level of innovation and explicit characterization for the conditions under which the optimal level of product degradation is 0 under a linear cost function, we also note that our model is more explicit about the ratio of intrinsic value and extrinsic value of the product. We also introduce a complementary good, which a firm can exploit by controlling the number of users it has.

**\section{Literature review}**

There exists a fairly rich literature on the pricing of network goods. The classic paper of networks in industrial organization by \citep{KS86} where it is shown that under competitive paradigm, firms don't have an incentive to make their product comptibility but they do have one to standardize. \cite{FT00} show that under a network good paradigm, the incumbent may decide to keep low prices even without a direct competitor as long as the threat of entry exists.

We also rely on the work for of \cite{N05} for our discussion on shaking hand equilibrium, whose work discusses cascade effects in a dynamic framework and makes the distinction between norm independent and norm dependent utilities and shows that an increase in the relative ratio of these two can induce cascade effects.

There is also work that specifically studies the effect of the quality differential between the legal and illegal copy of a pirated product. \cite{GL03} The differential is said to be quite small when it comes to music, whilst in software this gap is larger. Most piracy models do in fact assume perfect substitution between the two goods when, dropping this assumption does induce different pricing and profit outcomes.

Similarly \cite{PW06b} use a sampling model to show that giving free samples to consumers may be profitable for firms. \cite{C05} show that much of the potential benefits from piracy can be extracted by the firm if it employs a sampling strategy to draw in consumers and then sell it to them. However in most of the domains this paper discusses, this is likely not a pertinent strategy, mainly because the ability to create samples for your product is very likely correlated with the ability to differentiate it.

There is also work showing that in the digital space, with perfect copying, stronger copyright may act as a coordingating dervice between firms so that they may collude\citep{J08}. Indeed the work of \citep{S04} shows that prices are reduced in the presence of piracy, a result that is duplicated in this paper.

Perhaps the closets model to our own is the model by \cite{CRP91} where they also have an intermediate option of piracy. However their model does not include a product improvement variable and does not give an explicit solution for a specific distribution. Other models of piracy which mimic our approach are, \cite{MRSS17}, whom mimic our demand approach but focus more on the effects of piracy when open source competition is also a factor.

There is also work on the diffusion of innovation in more dynamic settings, where piracy is said to boost innovation during the early stages of the product lifecycle process. Studying this setting yields a result that is also found on this paper, mainly that strengthening piracy controls is not neccesarily an optimal strategy for digital markets. \citep{G03} \citep{GMM95}

**\section{setting up the model}**

In this workhorse model we linearly separate the products value into intrinsic valuation and extrinsic value of the product. The players in this model consist of a continuous spectrum of agents and a monopolistic firm. The firm knows the distribution of consumers valuations but but has limited price discriminatory powers.

We define $x\_i$ as the norm independent utility of consumer i associated to the consumption of the good whether it is pirated or bought.It is common knowledge that $x\_i$ is distributed according to F with an associated density \textit{f}. When an agent consumes they derive a socialization utility which depends on the taste for the good $x\_i$, the fraction of the population which consumes the good and a socialization parameter which may depend on whether the good has been bought, $\alpha$ or pirated $\beta$.

The action of the consumers consist of a choice between three discreet choices, to consume, to pirate and not to consume. The consumers are the standard rational agents who can calculate the future movement of all other consumers. Consumption and piracy both have a base value, c. The difference in these two choices in the basic model is that buying the product has a higher socialization slope, $\alpha$ which can be interpreted as more direct access to the network value of the product. Buying the good also has an added value, k a 'bonus material' that is added to the product. Pirating the good also yields the constant value $x\_i$ and a socialization value but the socialization value has a different slope,$\beta$. The difference between these two slopes can be interpreted as stigma from pirating, the representation of this is quite different but the motivation for this stigma is similar to \cite{CRP91}.

Pirating the good also has an additional cost, r which can be interpreted as product degradation or decrease in the probability of pirating or an increase in the penalty if pirating and caught. Finally the consumer has an opportunity cost of 0 where he can always decide not to consume.

The consumers maximization problem is the following:

\[

U\_i= \left\{

\begin{array}{ll}

x\_i(1+\alpha \left(E(1 - \int^{\overline{x}}\_{0}Q(t)f(t)dt)) \right) + k -p & if ~ he ~ buys ~ good \\

x\_i(1+\beta \left(E(1 - \int^{\overline{x}}\_{0}Q(t)f(t)dt)) \right) -r & if ~ he ~ pirates ~ good \\

0 & if ~ no ~ consumption \\

\end{array}

\right.

\]

We say that the distribution of x's is on the interval $[0,\overline{x}]$. The base value of the product, which is just the consumer valuation $x\_i$, this is the proportion of the value which consumers will benefit from regardless of whether they pirate or purchase. Note that the larger k is, the less important is the gap between the small valuation consumers and the high valuation.

The proportion of people who are in the network is given by $E(1 - \int^{\overline{x}}\_{0}Q(t)f(t)dt) $. Where $Q(t)$ is a monotonic function takes the value 1 when agents with valuation t not consumers and 0 if they are users. Similarly $G(t)$ is a function which takes the the value 1 when consumers are not buying and 0 otherwise. Since Q(t) includes both buying and pirating and G(t) is only buying. It follows that G(t) stochastically first order dominates Q(t).

Unlike the consumers, the monopolist has continuous choices which involve setting of two variables,the price,p, the degradation level of the pirated product,r. However a vital variable that affects these choices is the improvement of the bought product,k. Something to notice immediately is that an increase in the price does not have the same effect as an increase in the degradation level. We also assume that setting r=0 costs nothing to the firm.

In the literature, the only source of revenue for a firm is to sell the product, having piracy was only useful in that it motivated the buyers into their decision. However there are situations, especially in the digital world where having a larger user base also allows for additional direct profit opportunities. For instance in a social network, a larger user base allows the network owner to sell advertising slots at a higher price, and the wider the reach the more profitable the enterprise. We denote this exogenous source of revenue as $\lambda$. It should be noted that increasing pirates or users increase revenue from this source. The firms profit function is given by:

\begin{equation} \**label{eq:profit1}**

\pi(p,r,k)

=p(1-E(\int^{\overline{x}}\_{0}G(t)f(t)dt)) + \lambda (1-E(\int^{\overline{x}}\_{0}Q(t)f(t)dt))- c(r)

\end{equation}

Note that for the remainder of this paper, we will be referring to the segment of "users" as the proportion of the population that is either pirating or purchasing.

**\section{Resolving the model}**

**\subsection{Case where all agents are either users or non users}**

Note that if $\alpha = \beta$ we are in a case where only one type of using choice occurs. Where either no buying or no piracy is occurring. Users will only purchase if $k-p-r > 0$ and pirate otherwise, however the choice of whether to use or not use still depends on $x\_i$. If there are pirates then the cutoff will be $\check{x}$ and if there are buyers then $\tilde{x}$.

Since Q(t) is monotonic in t we have three cases: either $Q(t) = 1 \forall t \in [\underline{x},\overline{x}]$ where none of the agents are users. Either the $Q(t)=0 \forall t \in [\underline{x},\overline{x}]$, where all the agents are users. Or $\exists t \in [\underline{x},\overline{x}]$ where $Q(t+\epsilon)=0$ and $Q(t-\epsilon)=1$, where some agents are users and others are not.

In the first case where $Q(t) = 1 \forall t \in [\underline{x},\overline{x}]$, nobody is consuming. The implication is that for every consumer the choice to not consume is preferred. In this case no good is sold so revenue is 0. Since we assume that changing the price is cost-less and that setting $r = 0$ is cost-less this choice is strictly dominated from the firms point of view, as long as the interval is positive or $\lambda>0$.

In the second case where $Q(t)=0 \forall t \in [\underline{x},\overline{x}]$, we have three possible sub-cases. Either all consumers are pirates, all consumers are purchasers or there is mix of pirates and purchasers.

If all users are pirates the revenue of the enterprise is $\lambda$, the condition $ U\_p(x\_i)>U\_b(x\_i) \forall x \in [\underline{x},\overline{x}] $ must be met. From the firms point of view, the cost-less way of reaching this condition is to set r = 0 this condition is optimally satisfied by setting r to 0 and adjusting only p. In this case, profit will be equal to revenue, so profit will also be $\lambda$. This strategy is strictly dominated by a decreasing the price until an interval of consumers prefers to buy.

If all users are purchasers the revenue is $p+\lambda$ and the condition is that $ U\_p(x\_i)<U\_b(x\_i) \forall x \in [\underline{x},\overline{x}] $ . In this sub-case the conditions are $\underline{x}\geq \frac{p-r-k}{\alpha - \beta} $ and $\overline{x}\geq \frac{p-r-k}{\alpha - \beta} $. The Lagrangian gives us that $c'\_k=c'\_r=1$. So the level of the price depends on the cost functions, as long as the marginal cost for a fractional unit is less than unity then the profit in this scenario is strictly greater than the first sub-case.

Finally in the third sub-case it is $(1-E(\int^{\overline{x}}\_{\underline{x}}G(t)f(t)dt)) p +\lambda$. Where $(1-E(\int^{\overline{x}}\_{\underline{x}}G(t)f(t)dt))$ is the proportion of users who are buyers. This third sub-case implies that $\exists x\_i \in [\underline{x},\overline{x}]$ where $U\_p = U\_b$, we call this $x\_i = \hat{x}$, so $E(\int^{\overline{x}}\_{0}G(t)f(t)dt)$ = $E(\int^{\hat{x}}\_{\underline{x}}G(t)f(t)dt) +E(\int^{\overline{x}}\_{\hat{x}}G(t)f(t)dt)$. By definition the first term $G(t) = 1$ and the second term is 0. So we have $E(\int^{\hat{x}}\_{\underline{x}}f(t)dt)$, which is just the expectation of the CDF, $E(\int^{\hat{x}}\_{\underline{x}}f(t)dt)=E(F(\hat{x}(p,r,k)))=F(\hat{x}(p,r,k))$

\begin{proposition}

If $k>0$ and $\alpha \geq \beta$, no buyer will have a lower valuation than a pirate.

\end{proposition}

\begin{proof}

see appendix 3

\end{proof}

This is a standard assumption in the literature, higher valuation consumers will always be the ones purchasing, and only a lower valuation segment will be pirating. This is a reasonable assumption only if the bought product has a higher value than the pirated good. While our model follows this same route this may not necessarily always be true, most official streaming services have incomplete libraries whilst the pirate libraries are often exhaustive.

Here we can look to the firm, first note that in the case where pirates are on the higher end of the distribution and the case where buyers are on the higher end. The revenue differs only from the price the profit extracted from the complementary good is identical so the difference is in the setting of the price

**\subsection{Case where some agents are users and others are not}**

The implication of this scenario ($\exists x \in [\underline{x},\overline{x}]$ where $Q(t-\epsilon)=1$ and $Q(t+\epsilon)=0$) is that the firm has chosen some price and some level of deterrence where some consumers will not participate in consuming, whilst others will.

We denote by an identical argument as we have used for the derivation of the existence of $F(\hat{x}(p,r,k))$, we can denote the existence of an $\check{x}$ which denotes the point at which Q(t) changes value implies a CDF, $F(\check{x}(r))$. We also denote the point at which $F(\check{x}(r))=F(\hat{x}(p,r,k))$ as $\tilde{x}$ and its corresponding value on the distribution as $F(\tilde{x}(p,r,k))$.

We can interpret the cutoff point $\check{x}$ as the consumer who is indifferent between using and not using. Similarly $\tilde{x}$ is the cutoff point for buying and not using when no pirates exist. To establish which portion of consumers is pirating we need only compute $F(\hat{x})-F(\check{x})$ similarly, the proportion buying is given by $F(\overline{x})-F(\hat{x})=1-F(\hat{x})$. So the number of pirates exceeds the number of buyers if $F(\hat{x})-F(\check{x}) \geq F(\overline{x})-F(\hat{x})$.

This case can give rise to 3 sub-cases, either only pirates and non-users exist, either only buyers and non-users or all three types exist.

If only pirates and non-users exist, then the revenue of the enterprise is $\lambda(1-F(\check{x}(r)))$. This sub-case is strictly dominated by the a pirates only case because the enterprise can reach revenue $\lambda$ by decreasing r, which increases revenue and reduces cost.

If buyers and non-users exist then revenue is $p(1-F(\tilde{x}(p,r,k)))+\lambda(1-F(\tilde{x}(p,r,k)))$.

Finally if all three types exist then our profit function takes the following form.

\begin{equation} \**label{eq:profit1}**

\pi(p,r,k)

=p(1-F(\hat{x}(r,p,k))) + \lambda (1-F( \check{x}(r) ))- c(r)

\end{equation}

Notice the price is only multiplied by the $1 - F(\hat{x})$ term, this is because not all of the $1- F(\check{x})$ users will be paying for the product. Nevertheless, as seen above, $\hat{x}$ depends on $\check{x}$, so a change in $\check{x}$ changes both the first and second term. The lower indifference condition still has an indirect effect on the level of profit. That is the firm has to take care because if it sets the price too high then the consumers will just shift to pirating. Plugging in the indifference conditions and optimizing we can acquire the price(see appendix 2 for details). The $\lambda$ is the complementary good or revenue which is attained by virtue of having a larger network. This may be a complementary good that users will want to buy or it may be ad revenues that the company can acquire, where a larger user base increases the potential revenue.

**\subsection{Demand functions and conditions}**

We now compute explicit demand functions and the three indifference conditions.

We have three cutoff, points but only up to 2 can be used at one time. The consumer who is indifferent between pirating and not consuming yields an indifference condition of:

\[

x\_i + x\_i\beta \left(1-F(\check{x})\right) -r = 0

\]

\begin{equation} \**label{eq:indi1}**

\Rightarrow \check{x}(1 + \beta(1-F(\check{x}))) = r

\end{equation}

Similarly, the consumer who is indifferent between buying and pirating has a valuation of:

\[

x\_i + x\_i\alpha \left( 1-F(\check{x}) \right) + k -p = x\_i + x\_i\beta \left(1-F(\check{x})\right) -r

\]

\begin{equation} \**label{eq:indi2}**

\Rightarrow \hat{x} = \frac{p-r-k}{(\alpha - \beta)(1 - F(\check{x}))}

\end{equation}

Finally, the consumer who is indifferent between buying and not consuming is indifferent between

\[

x\_i + x\_i\alpha \left( 1-F(\check{x}) \right) + k -p = 0

\]

\begin{equation} \**label{eq:indi3}**

\Rightarrow \tilde{x} = \frac{p-k}{1 + \alpha(1 - F(\check{x}))}

\end{equation}

It is trivial that if the value added, k, is 0, and the price is greater than the degradation, $p\geq r$, then people will only purchase if $\alpha$ is greater than $\beta$.

Up until now we have not assumed a specific distribution for the willingness to pay of consumers. We now adopt the uniform distribution to get explicit functions for our indifference conditions and derive our demand for each consumer choice.

\begin{proposition}

%\label{Price}

When consumer tastes are distributed uniformly between $0$ and $\overline{x}$; The proportion of people who are users and consuming, respectively, are denoted by:

\begin{equation}\**label{eq:1}**

1 - F(\check{x}) =\frac{ \beta - 1 \mp \sqrt{ (1+\beta)^{2}- \frac{4 r \beta}{\overline{x}} }}{2 \beta }

\end{equation}

\begin{equation}\**label{eq:2}**

1 - F(\hat{x})=1-\frac{(p-r-k)2 \beta}{(\alpha - \beta) \left( \beta - 1 \mp \sqrt{ (1+\beta)^{2}- \frac{4 r \beta}{\overline{x}} }\right) \overline{x} }

\end{equation}

\begin{equation}

1-F(\tilde{x}) = 1-\frac{2 \beta(p-k)}{\overline{x}(2 \beta + \alpha \beta - \alpha -\alpha j) }

\end{equation}

\end{proposition}

\begin{proof}

See appendix 1

\end{proof}

\begin{corollary}

When $\beta=1$ these expressions reduce to:

\begin{equation}\**label{eq:3}**

1 - F(\check{x}) =\mp \sqrt{1-\frac{r}{\overline{x}}}

\end{equation}

\begin{equation}\**label{eq:4}**

1 - F(\hat{x})= 1 - \frac{p-r-k}{\overline{x}(\alpha - 1) \mp \sqrt{\frac{\overline{x}-r}{\overline{x}}} }

\end{equation}

\begin{equation}\**label{dtilde}**

1-F(\tilde{x}) = 1-\frac{ (p-k)}{\overline{x}(1 \mp \alpha \sqrt{\frac{\overline{x}-r}{\overline{x}}} ) }

\end{equation}

\end{corollary}

Since only one hundred per cent of consumers can take any one action, demand functions are bounded at 1. Note that if r is 0, equation \ref{eq:3} reaches unity, on the other hand, other values of . Similarly, if p is equal to 0, and we know that r,k $\geq 0$ and $\alpha>\beta$ then the demand function is 1. and k are equal to 0 and beta is positive, equation \ref{eq:4} pirate expression reduces to 1.

The different functions of this model derived in detail in appendix 1 and 2:

Demand for buying if pirates:

$D\_b = 1 - F(\hat{x}) = 1 - \frac{p-r-k}{(\alpha - 1) (\mp \sqrt{1-\frac{r}{\overline{x}}}) \overline{x} }$

Demand for buying if no pirates: $ \hat{D}\_b =1-F(\tilde{x}) = 1-\frac{p-k}{\overline{x}(1 \mp \alpha \sqrt{1-\frac{r}{\overline{x}}}) }$

Demand for using: $D\_u = 1 - F(\check{x}) = \mp \sqrt{1-\frac{r}{\overline{x}}}$

Demand for piracy: $D\_p = F(\hat{x})-F(\check{x})= \frac{p-r-k}{(\alpha - 1) ( \mp \sqrt{1-\frac{r}{\overline{x}}}) \overline{x}} - 1 \mp \sqrt{1-\frac{r}{\overline{x}}}$

Demand for not using: $D\_0 = F(\check{x})=1 \mp \sqrt{1-\frac{r}{\overline{x}}}$

\begin{proposition}

The incentive to improve the bought product is higher in the presence of piracy.

\end{proposition}

\begin{proof}

We want to compare the effect of k on demand to buy in the presence of pirates with its effect on demand to buy without the presence of pirates. First note that the numerator relevant to k is identical in both \ref{eq:4} and \ref{dtilde}. So if the denominator in \ref{eq:4} is smaller than the one in \ref{dtilde}, then we will have shown that an increase in k has a greater effect on demand in the case with piracy. $(\alpha-\beta)\overline{x}>\overline{x}(2\beta+\alpha \beta -\alpha -\alpha j) \rightarrow j-\beta -1>0$. These terms are all locally negative so the LHS is smaller.

\end{proof}

With these conditions in hand we can now summarize the above analysis for the different cases. As a frame of reference we will be using the demand functions for piracy and the demand function for not using, these two conditions are sufficient to represent the possible cases. In red we see the cases which are dominated by the firm.

\begin{tabular}{ | l | l | l | l | l |}

\hline

$D\_p >0$ & $D\_0>0$ & $D\_p + D\_0 < 1$ & Active types & Profit \\ \hline

Binding & Not binding & Not binding & Buyers and nothing& $p(1-F(\tilde{x}))+\lambda(1-F(\tilde{x}))$ \\ \hline

Binding & Not binding & Binding & Nothing& \textcolor{red}{0} \\ \hline

Binding & Binding & Not binding & Buyers only & $p + \lambda$ \\ \hline

Binding & Binding & Binding & Impossible or 0 Mass & N/A \\ \hline

Not binding & Not binding & Not binding & Three Segments exist & $p(1-F(\hat{x}))+\lambda(1-F(\check{x}))$ \\ \hline

Not binding & Not binding & Binding & Pirates and nothing & \textcolor{red}{$p(1-F(\check{x}))+\lambda(1-F(\check{x}))$} \\ \hline

Not binding & Binding & Not binding & Pirates and buyers & $p(1-F(\hat{x}))+\lambda$ \\ \hline

Not binding & Binding & Binding & Pirates only. & \textcolor{red}{$\lambda$} \\ \hline

\end{tabular}

\begin{proposition}

%\label{Price}

When consumer tastes are distributed uniformly. The monopolists optimal price is

\begin{equation} \**label{eq:price}**

p\_m = \frac{1}{2}\left(r+k + \frac{(\alpha - \beta)\left( \beta -1 \mp \sqrt{ (1+\beta)^{2}- \frac{4 r \beta}{\overline{x}} } \right)}{2 \beta} \right)

\end{equation}

\end{proposition}

\begin{proof} **\label{eq:price}**

See appendix 2

\end{proof}

\begin{corollary}

When $\beta=1$ price reduces to the following.

\begin{equation}

p\_m = \frac{1}{2}\left(r+k + (\alpha - 1) \left( \mp \sqrt{ \frac{\overline{x}- r }{\overline{x}} } \right)\right)

\end{equation}

\end{corollary}

Note here that the value inside the square root will necessarily be between 0 and 1 due to the fact that $r<\overline{x}$. So unless $\alpha > 2$, the third term will be small.

\begin{observation}

Higher degradation increases the monopolists optimal price.

\end{observation}

\begin{observation}

Higher values of bonus material increase the price.

\end{observation}

The price is intuitive, the higher the gap between the two slopes in the utilities, $\alpha - \beta$, the higher the possible rent that can be achieved. Both r and k increase the price level, but k in a linear fashion whilst increases in r increases price with diminishing effects but at least as potently as k. The upper bound has a weak effect on price unless the externality gap between piracy and buying is very large.

After plugging in the optimal price into the profit function and simplifying we acquire the following expression(see appendix 3):

\begin{proposition}

%\label{Profit symmetry}

The if the cost of product improvement has a linear form, the firms optimal level is denoted by:

\begin{equation} \**label{eq:optimalk}**

k = \frac{ \overline{x}}{\beta}(\alpha - \beta) \left( \beta - 1 +\sqrt{(1+\beta)^2 -\frac{4r\beta}{\overline{x}}}\right) \left( \frac{1}{2 \beta} - c \right) - r

\end{equation}

\end{proposition}

\begin{proof}

(see appendix 6)

\end{proof}

Note that the the second parenthesis is always positive.

\begin{observation}

Higher levels of r decreases the optimal k.

\end{observation}

\begin{observation}

Higher $beta$ decreases the optimal k

\end{observation}

\begin{observation}

Higher $\alpha$ increases the optimal k

\end{observation}

These observations also hold if the cost of k is quadratic, this is shown in appendix 5.

**\subsubsection{Optimal product degradation}**

In this section we will assume $\beta=1$. Product degradation is exogenous to the firm. Using the optimal conditions $k^\*$ and $r^\*$ we can use the envelope theorem to make the following proposition.

\begin{proposition}

The profit maximizing level of product degradation is 0 in the case where the cost function of of product improvement is linear as long as the cost is less than 0.5.

\end{proposition}

\begin{proof}

see appendix 6

\end{proof}

\textcolor{blue}{needs text here}

What makes this network good paradigm so interesting is that in the presence of specific uncertainties about the exact valuation of consumers, it is a challenge for the firm to completely extract the value its product provides. While in an independent utilities framework it can more or less guarantee a certain revenue by setting a certain price, as utilities become more and more correlated, the firm loses its ability to accurately extract value. While in normal menu pricing, the effect of pricing consumers out of the markets are only direct, in this framework the effects are much more indirect as they shrink the size of the population consuming. If the are uncertainties about the relative ratio of norm dependent and norm independent utilities then the choice of the firm becomes asymmetric. Depending on the level of this uncertainty, a firm may wish to place r lower, as if it places the r too high it risks losing all of its profits, whilst if it places it too low it merely loses a small amount relative to its optimum. In other words, this kind of equilibrium is not a trembling hand equilibrium, imperfectly placing these variables does have cascade effects.

**\section{Discussion}**

We have attempted to model the incentives of the firm in the face of piracy and product improvement. Both of these have different effect on profits depending on parameter values. In practice, firms likely control the ability to control their products much more than the ability to chase after pirates. Chasing after pirates is often left up to government agencies with very little input from individual firms. Our model shows that even if the firms had total control of the level of piracy pursuit, they would not necessarily fully pursue pirates this.

The intuition behind this carrot and stick approach is that the price has the effect of simply making buyers switch to piracy, whilst increasing the product degradation both increases the proportion of people buying but also decreases the user base. This decrease in the user base corresponds to the product becoming less popular and the increase in buying represents a consumer who no longer wishes to undertake the risk of pirating.

The level of product improvement a firm will choose to pursue will depend on whether pirates exist. The presence of pirates acts as a competitive force which pushes the firm to improve its bought product. Clearly if the product degradation is high there will be less incentive to improve the product so if the firm controls the level of piracy and the cost of improving is too high, it will favor increasing r.

It is according to this model, insufficient to argue based on the number of purchasers that a firm profits will be affected, because the valuation and the number of purchasers are not independent variables. Indeed a change of structure for the industry in the presence of piracy will not necessarily decrease profits.

Protection in this paradigm is not endogenized however the condition derived in this model is a sufficient condition for not wishing to set the cost of piracy above 0. A DRM paradigm, can be interpreted as merely an increase in r as it decreases the probability of consuming the product \citep{S04}.

Note that in our analysis $\alpha$ and $\beta$ also represent the proportion of value of which the externality represents. When we fix the value of $\beta$ to 1 we are in fact saying that the ratio of norm dependent to norm independent utility is 1.

The model we employ here is static, however our conclusions can intuitively be extended to some dynamic cases. If we consider that the firm sets its variables and then the consumers continually make choices every period some results may change. Perhaps most importantly, in a continuum of consumers, though an individual consumers making the wrong choice has no impact on the equilibrium. If the firm makes a slightly wrong choice in p this will not cause a cascade, but if r is set slightly wrong, this can cause a cascade effect. That is, if the optimal $r$ is $r^\*$ and the firm sets $r^\*+\epsilon$, then there will be some interval of pirates who will no longer find it optimal to pirate and will drop out, this dropping out will reduce the utility of some other consumers and the effect will continue to cascade. This cascade effect will be reducing the proportion of the value which is norm dependent.

As pointed out in \cite{CRP91} this setup raises strange ethical questions. When we have possible parameter values that imply that higher pursuit of pirates may be worse for consumers and producers as well as the aggregate society which must incur the cost of enforcing this policy.

Further work: It is unclear if in a dynamic setting, the firm would attempt to use piracy at all, or at least this would not be a trembling hand equilibrium because

This model has assumed the existence of the network good, a further question to ask is how the presence of piracy may change the incentives to pirate.

Further work:

While the assumption that people will be mixing with people of all valuations does not seem unreasonable, in the real world there would be a higher probability of mixing with people of ones own valuation. So a more realistic model would be one where people give higher value to those who are closer and lower to those who are further. In such a situation, the effects described in this paper would clearly be weaker. However, there are other structures, such as families, where the mixing is much more random and hence the proximity representation may be not be adequate. It is unclear if self segregation in a social value environment increases or decreases profits, though if the choice is between no socialization value and some, clearly the firms prefer the latter.

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The key variables in this model are r, the level of product degradation and the form of product improvement. Firms may have incentives to favor one over the other depending on their cost functions. The intuitive story here is that a firm may have the choice to increase its profit by either limiting the people who consume its product or by finding ways to add value to its buying customers. Trivially, if a firm cannot create the product in such a way where it can offer at least two types of services, then it has no incentive to let consumers realize their socialization value. }

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The cost function of r variable is key to setting it at the appropriate level from a profit maximizing perspective. As mentioned earlier, if the cost function is not convex the r will be set at the level $\hat{x}=\check{x}$ which directly maximizes profit. One interpretation of r is the consumers inconvenience in pirating the product, another is the expected loss of being caught and penalized, either way in a world with intellectual property, this cost is not borne by the company. On the contrary this cost is often borne by the government. In the second interpretation the government is the one with the capability of punishing the use of the product, so there is an incentive by the company to set r to the highest level possible. }